

Pairs of chiral quarks on the lattice from staggered fermions

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Abstract

A new formulation of chiral fermions on the lattice is presented. It is a version of overlap fermions, but built from the computationally efficient staggered fermions rather than the previously used Wilson fermions. The construction reduces the four quark flavors described by the staggered fermion to two quark flavors; this pair can be taken as the up and down quarks in Lattice QCD. A domain wall formulation giving a truncation of this overlap construction is also outlined.

Keywords: Lattice QCD, chiral fermions, staggered fermions

1. Introduction

It is of great interest to find lattice formulations of Quantum Chromodynamics (QCD) with exact chiral symmetry. In such a formulation, spontaneous chiral symmetry breaking and the associated spectrum of Goldstone bosons (i.e. the light mesons), as well as the emergence of the flavor-singlet (η' meson) mass via the index theorem connection between quark zero-modes and gauge field topology [1], can be studied directly in the lattice model without the usual need to extrapolate via chiral perturbation theory. Finding such formulations is problematic though [2], and was a long-standing challenge in Lattice QCD. An explicit formulation of this kind was finally found by Neuberger [3] via the overlap formulation of chiral fermions on the lattice [4]. It is a chirally improved version of Wilson fermions, and realizes exact chiral symmetry of the Ginsparg-Wilson (GW) type [5, 6, 7]. However, its theoretical attractiveness is offset in practice by the high computational cost of implementing it in simulations.

In this paper, building on insights from [8], a new version of overlap fermions is presented constructed from the staggered lattice fermion rather than the previously used Wilson fermion. The staggered fermion is the most computationally efficient lattice fermion formulation known, and this advantage is expected to be inherited by the new staggered version of overlap fermions. The challenge in doing this is that the overlap construction relies on specific properties of Wilson fermions that are not shared by staggered fermions. This challenge is overcome here using a new theoretical idea of interest in its own right: a Wilson-like phase of the staggered fermion theory can be created by adding a certain “flavored” mass term. In this new phase, the number of quark flavors described by the staggered fermion is reduced from four to two, and the staggered fermion acquires Wilson-like properties. The latter allow the overlap construction to work as desired. The resulting staggered overlap fermion describes two quark flavors and has an exact *unflavored* GW chiral symmetry originating from the exact flavored chiral symmetry of the original staggered fermion.

However, the *flavored* vector and chiral symmetries of the quark pair described by the staggered overlap fermion are not exact since the corresponding symmetries of the original staggered fermion are broken by lattice effects. Therefore, exact flavored vector and chiral symmetries only hold for *pairs* of quark flavors when each pair is described by a staggered overlap fermion.

The two quark flavors described by the staggered overlap fermion can be taken as the almost massless u and d quarks in QCD. Then, due to the exact unflavored GW chiral symmetry, this description has all the advantages of the usual overlap fermions regarding the axial $U(1)$ anomaly and index theorem connection between quark zero-modes and gauge field topology. This is connected with some of the most subtle and interesting parts of QCD physics – not only the η' mass mentioned already but also the order of the QCD phase transition at finite temperature – which are also computationally highly demanding to study in lattice QCD (see, e.g., [9]). Finding a new lattice approach in which these features are not distorted by lattice effects, and which is computationally more efficient than usual overlap fermions, is therefore significant. As an indication of the desirability of this, it can be noted that high-precision calculation of the η' mass is still an outstanding problem in Lattice QCD – the latest state of the art calculation, using Wilson-based domain wall fermions (a truncated version of usual overlap fermions), has an uncertainty of 15% [10].

2. Staggered overlap fermion construction

The staggered lattice fermion field is a one component Grassmann field $\chi(x)$, $x = an \in a\mathbb{Z}^4$, describing 4 degenerate Dirac fermions. The staggered Dirac operator acting on these fields is $D_{st} = \eta_\mu \nabla_\mu$ where $\eta_\mu \chi(x) = (-1)^{n_1 + \dots + n_{\mu-1}} \chi(x)$ and $\nabla_\mu = \frac{1}{2a}(T_{+\mu} - T_{-\mu})$, with the parallel transporters $T_{\pm\mu}$ given in terms of the lattice link variables $U_\mu(x)$ by $T_{+\mu}\chi(x) = U_\mu(x)\chi(x+a\hat{\mu})$ and $T_{-\mu} = (T_{+\mu})^{-1}$. There is an exact *flavored* chiral symmetry

$$\{\Gamma_{55}, D_{st}\} = 0 \quad (1)$$

where Γ_{55} , given by $\Gamma_{55}\chi(x) = (-1)^{n_1 + \dots + n_4}\chi(x)$, corresponds to $\gamma_5 \otimes \gamma_5$ in the spin-flavor interpretation of staggered fermions [11]. The first and second γ_5 factors act in spinor and flavor space, respectively. Note the properties $\Gamma_{55}^2 = \mathbf{1}$ and $\Gamma_{55}^\dagger = \Gamma_{55}$.

Direct application of the overlap construction [3] to staggered fermions by simply replacing the kernel operator $D_W - M$ (where D_W is the Wilson-Dirac operator) by $D_{st} - M$ does not give anything useful: The exact flavored chiral symmetry of the staggered fermion is lost, there is no GW symmetry to replace it, and there are no exact zero-modes in general. However, as we will now show, the situation changes if, instead of a scalar mass M , we use the following “flavor-chiral” mass term:

$$M_{st} = \frac{r}{a} \Gamma_{55} \Gamma_5 \quad , \quad r > 0 \quad (2)$$

Here Γ_5 is the staggered fermion version of the chirality matrix, corresponding to $\gamma_5 \otimes \mathbf{1}$ up to $O(a^2)$ discretization errors in the spin-flavor interpretation [11]. Consequently, in the spin-flavor interpretation,

$$M_{st} \sim \mathbf{1} \otimes \gamma_5 + O(a). \quad (3)$$

Explicitly, Γ_5 is given as follows. Use the parallel transporters to define $C_\mu = \frac{1}{2}(T_{+\mu} + T_{-\mu})$, and let $C = (C_1 C_2 C_3 C_4)_{sym}$ denote the symmetrized product of C_1, C_2, C_3, C_4 . Then $\Gamma_5 = \eta_5 C$ where $\eta_5 = \eta_1 \eta_2 \eta_3 \eta_4$. (Explicitly, $\eta_5 \chi(x) = (-1)^{n_1 + n_3} \chi(x)$.) Note that Γ_5 is hermitian and commutes with Γ_{55} ; therefore M_{st} is hermitian. The effect of this mass term in

$$D_{st} - M_{st} \quad (4)$$

is to split the low-lying modes into branches with approximately definite positive and negative flavor-chirality, giving them masses $-\frac{r}{a}$ and $\frac{r}{a}$, respectively.

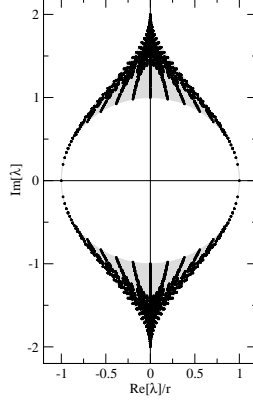


Figure 1: Spectrum of $D_{st} - M_{st}$ in the free field case [12].

This is shown in the free field case in Fig. 1, taken from [12]. In this case, there are 16 zero-modes of D_{st} with momenta $p_A = \frac{\pi}{a}A$, $A_\mu \in \{0, 1\}$. On the vectorspace V spanned by these modes we have $(\Gamma_{55}\Gamma_5)^2 = \mathbf{1}$, so there is a decomposition $V = V_+ \oplus V_-$ with $\Gamma_{55}\Gamma_5 = \pm \mathbf{1}$ on V_\pm , and it can be shown that $\dim(V_\pm) = 8$. Hence $D_{st} - M_{st} = \mp \frac{r}{a}$ on V_\pm ; this shows how the real eigenvalues arise in the free field case. Moreover, the operators $\eta_\mu T_{+\mu}$ and $\eta_\mu T_{-\mu}$ making up D_{st} coincide on V and can be shown to commute with $\Gamma_{55}\Gamma_5$ on V . Hence their representations Γ_μ on V (given in [11] by the 16×16 matrices $(\Gamma_\mu)_{AB}$), which form a representation of the Euclidean Dirac algebra [11], decompose into representations Γ_μ^\pm on V_\pm . From this it can be seen that, from a low momentum viewpoint, the 4 Dirac fermion species described by the staggered fermion are split into two pairs of Dirac fermions, with the pair with positive (negative) flavor-chirality having mass $-\frac{r}{a}$ ($+\frac{r}{a}$).

Upon inserting the kernel $D_{st} - M_{st}$ into the overlap formula,

$$D_{so} = \frac{r}{a} \left(1 + (D_{st} - M_{st}) \frac{1}{\sqrt{(D_{st} - M_{st})^\dagger (D_{st} - M_{st})}} \right), \quad (5)$$

the negative mass modes are converted into the physical modes of a massless fermion, while the positive mass modes are converted into modes with heavy masses $\sim 1/a$ which decouple in the continuum limit, just as in the Wilson case. Thus only two of the original four quark flavors survive in the resulting overlap fermion. Furthermore, since the physical modes have approximately

definite positive flavor-chirality, $\Gamma_5 = \Gamma_{55}$ on these modes up to $O(a^2)$ effects.

The staggered overlap Dirac operator (5) satisfies the following GW relation:

$$\{\Gamma_{55}, D_{so}\} = \frac{a}{r} D_{so} \Gamma_{55} D_{so} \quad (6)$$

Note that this is now an *unflavored* chiral symmetry, since, as remarked, $\Gamma_{55} = \Gamma_5 + O(a^2)$ on the physical modes of the staggered overlap fermion. The GW relation can be readily verified after noting that the staggered overlap operator can be expressed as

$$D_{so} = \frac{r}{a} \left(1 + \Gamma_{55} \frac{H_{st}}{\sqrt{H_{st}^2}} \right) \quad (7)$$

with the hermitian operator H_{st} given by

$$H_{st} = \Gamma_{55}(D_{st} - M_{st}) = \Gamma_{55}D_{st} - \frac{r}{a}\Gamma_5 \quad (8)$$

It also follows from (7) that $D_{so}^\dagger = \Gamma_{55}D_{so}\Gamma_{55}$, which implies that the non-real eigenvalues of D_{so} come in complex conjugate pairs. Since the only possible real eigenvalues (which must lie on the GW circle) are 0 and $2r/a$, it follows that the fermion determinant $\det(D_{so})$ is real and non-negative, as required for lattice QCD simulations.

A required property of the lattice Dirac operator is locality. It holds for D_{so} ; this can be proved analytically by the same argument as in the Wilson overlap case [14], using the staggered bound mentioned in [8], when the lattice gauge fields satisfy an admissibility condition; the details are in [13]. Locality of D_{so} in realistic gauge field backgrounds has been shown numerically in [12].

Computational efficiency of D_{so} was also studied in [12]: preliminary numerical evidence indicates a speedup of order 2-3 compared to the Wilson-based overlap Dirac operator in realistic gauge field backgrounds.

3. Pairs of exact zero-modes and index theorem

We now verify that D_{so} has pairs of exact chiral zero modes with index determined by gauge field topology in accordance with the 2-flavor index theorem when the lattice gauge field background is not too rough. Due to the GW relation (6), the index is well-defined as $\text{index}(D_{so}) = n_+ - n_-$

where n_{\pm} is the number of independent zero-modes of D_{so} with \pm chirality under Γ_{55} (which is the same as Γ_5 on these modes up to $O(a^2)$ effects). A general formula [6] gives

$$\text{index}(D_{so}) = -\frac{1}{2}\text{Tr}\left(\frac{H_{st}}{\sqrt{H_{st}^2}}\right). \quad (9)$$

We connect this to the hermitian staggered operator studied previously in [8], denoted by \tilde{H}_{st} below:

Theorem. *For every value of m , the hermitian staggered operators*

$$H_{st}(m) = \Gamma_{55}D_{st} - m\Gamma_5, \quad \tilde{H}_{st}(m) = iD_{st} - m\Gamma_5$$

have the same eigenvalue spectrum.

The proof of the theorem is deferred to [13]. It can be understood intuitively from the fact that $\tilde{H}_{st}(m)$ arises from $H_{st}(m)$ by a change of representation of the staggered sign factors $\eta_{\mu} \rightarrow i\Gamma_{55}\eta_{\mu}$ in D_{st} .

In light of the theorem we can replace H_{st} in (9) by $\tilde{H}_{st}(m)$ with $m = \frac{r}{a}$; here r is the parameter in (2). But then the results of [8] give

$$\text{index}(D_{so}) = \frac{1}{2}\text{index}(D_{st}) = 2Q \quad (10)$$

in sufficiently smooth gauge field backgrounds of topological charge Q and with $\frac{r}{a}$ lying in a suitable range. The last equality in (10) was confirmed in the numerical study in [8].

4. Stability of the massless phase under radiative corrections

It is important to consider the symmetries of the staggered overlap fermion since these need to be sufficient for renormalizability and stability of the massless 2-flavor phase under radiative corrections. Besides the replacement of the flavored chiral symmetry (1) by the GW symmetry (6), the usual staggered fermion symmetries listed in [11] all hold for the staggered overlap fermion, except for axis reversals and shift transformations. Under the latter $\bar{\chi}\Gamma_{55}\Gamma_5\chi \rightarrow -\bar{\chi}\Gamma_{55}\Gamma_5\chi$, and therefore $\bar{\chi}D_{so}\chi$ is invariant when these transformations are combined with the parameter flip $r \rightarrow -r$ in (2). The same invariances must hold for all counterterms that arise. Hence the counterterms must be invariant under all the usual staggered fermion symmetries except (1), and except for a possible sign change under axis reversals and shift transformations. It is easy to show [13] that the most general *local* mass-dimension

4 terms with these properties are (up to proportionality, and modulo $O(a)$ terms)¹

$$\bar{\chi} D_{st} \chi, \quad \bar{\chi} \{\Gamma_{55} \Gamma_5, D_{st}\} \chi, \quad \frac{1}{a} \bar{\chi} \Gamma_{55} \Gamma_5 \chi, \quad \frac{1}{a} \bar{\chi} \chi \quad (11)$$

Since D_{so} is local, all counterterms are also local, and the theory is renormalizable, provided the power-counting theorem holds [15]. It holds in the Wilson overlap case [16] and is plausible in the present case. Assuming this, stability can be established to all orders in the loop expansion as follows. The GW relation (6) leads to

$$\Gamma_{55} \langle D_{so}^{-1}(q) \rangle + \langle D_{so}^{-1}(q) \rangle \Gamma_{55} = \frac{a}{r} \Gamma_{55} \quad (12)$$

where $\langle D_{so}^{-1}(q) \rangle = (D_{so}(q) + \Sigma(q))^{-1}$ with $\Sigma(q)$ being the fermion self-energy. Here and in the following, $X(q)$ denotes the free field momentum representation of the staggered operator X ; it is a linear map (16×16 matrix) $X(q) : V \rightarrow V$ on the zero-mode vector space V defined earlier, with $q_\mu \in [-\frac{\pi}{2a}, \frac{\pi}{2a}]$. The momentum representation of Γ_{55} is a constant matrix, and $\Gamma_{55} \Gamma_5(q) = \pm C(aq) \mathbf{1}$ on V_\pm where $C(aq) = \prod_\mu \cos(aq_\mu) = 1 + O(a^2 q^2)$.

In light of the preceding, at one loop $\Sigma(q)$ has the form

$$c_1 D_{st}(q) + c_2 \Gamma_{55} \Gamma_5(q) D_{st}(q) + \frac{c_3}{a} \Gamma_{55} \Gamma_5(q) + \frac{c_4}{a} \quad (13)$$

up to irrelevant terms, with each $c_j = c_j(a^2 q^2)$ diverging no worse than logarithmically for $a \rightarrow 0$. Note that all terms in (13) map V_\pm to itself, as does $D_{so}(q)$. From (12)–(13) and the fact that $D_{so}(0) = 0$ on V_+ it is straightforward to show $\Sigma(0) = 0$ on V_+ . (We omit the details; an analogous result was found in the Wilson overlap case in [16, 17]). This implies $c_4 = -c_3$ in (13), so on V_+ the last two terms in (13) combine to give an irrelevant term $\frac{c_3}{a}(C(aq) - 1)$ which vanishes $\sim a \log(a)$ for $a \rightarrow 0$. The first two terms in (13) act on V_\pm as $(c_1 \pm c_2 C(aq))(\Gamma_\mu^\pm \frac{i}{a} \sin(aq_\mu))$. It follows that, for $a \rightarrow 0$, $D_{so}(q) + \Sigma(q) = \Gamma_\mu^+ i q_\mu (1 + c_1(a^2 q^2) + c_2(a^2 q^2))$ on V_+ . The self-energy is hereby seen to have the usual effect of a logarithmically divergent wave function renormalization on the massless physical modes. Then, after renormalizing the theory at one loop, the argument above can be repeated to get the same result at two loops, and so on. In this way stability is seen to hold to all orders in the renormalized theory.

¹This is with the lattice gauge fields included. In the free field case $\Gamma_{55} \Gamma_5$ and D_{st} commute and the second term in (11) simplifies.

5. Wilson-like nature of the staggered overlap fermion kernel

A deeper understanding of the staggered overlap construction can be obtained by clarifying the Wilson-like nature of its staggered fermion kernel. To this end we introduce the following 2-flavor Wilson-like version of the staggered Dirac operator:

$$D_{sW} = D_{st} + W_{st} \quad , \quad W_{st} = \frac{r}{a}(1 - \Gamma_{55}\Gamma_5). \quad (14)$$

The term W_{st} plays an analogous role to the Wilson term in the Wilson-Dirac operator: it decouples the negative flavor-chirality modes by giving them mass $2r/a$ while keeping the two positive flavor-chirality modes as the physical modes. Hence D_{sW} describes two physical quark flavors on which $\Gamma_5 = \Gamma_{55}$ up to $O(a)$ effects. In light of this we can obtain a 2-flavor overlap fermion by taking $D_{sW} - M$ with $M = r\rho/a$, $\rho \in (0, 2)$ as the kernel in the usual overlap construction. For $\rho = 1$ this is precisely the 2-flavor staggered overlap Dirac operator D_{so} we constructed above in (5), since $D_{sW} - \frac{r}{a} = D_{st} - M_{st}$. But now we see that it can be generalized to any $\rho \in (0, 2)$. Furthermore, the role of the parameter r in the staggered overlap construction is hereby clarified: it is analogous to the Wilson parameter in the usual overlap construction based on Wilson fermions. This is surprising, since r/a in (2) initially appears to be analogous to the mass parameter $M = \rho/a$ in the Wilson case.

Starting from Wilson fermions, a domain wall fermion formulation can be constructed [18] which gives a truncation of the overlap fermion construction [19]. The same can now be done with staggered fermions simply by replacing

$$D_W \rightarrow D_{sW} \quad , \quad \gamma_5 \rightarrow \Gamma_{55} \quad (15)$$

in the previous Wilson-based constructions. The lattice Dirac operator for the staggered domain wall fermion in 5 dimensions is then

$$D_{sdw} = D_{sW} - M + \Gamma_{55}\tilde{\partial}_s \quad , \quad M = \frac{r\rho}{a} \quad , \quad \rho \in (0, 2) \quad (16)$$

where $s \in [0, L]$ is the lattice coordinate of the 5th dimension and $\tilde{\partial}_s = P_+\partial_s^{(+)} + P_-\partial_s^{(-)}$ with $P_\pm = \frac{1}{2}(1 \pm \Gamma_{55})$ and $\partial_s^{(+)}$, $\partial_s^{(-)}$ being the forward and backward finite difference operators.² Boundary conditions on $\chi(x, s)$

²This is completely unrelated to a previous proposal for staggered domain wall fermions in [20] which describes 4 rather than 2 flavors.

at $s = 0$ and $s = L$ are introduced analogously to the Wilson-based case; then the staggered domain wall construction is seen to be a truncation of the staggered overlap construction by essentially the same argument as in [19]. The details will be given in [13].

6. Concluding discussion

Underlying the staggered overlap and staggered domain wall fermion constructions in this paper is a staggered version of Wilson fermions obtained by adding a flavored mass term to the staggered fermion action. The idea of considering staggered fermion with flavored mass term is not new in itself; this was already done many years ago by Golterman and Smit in [11]. Rather, the new idea here is that, for a certain choice of flavored mass term, namely the Wilson term W_{st} in (14) determined by the “flavor-chiral” mass term M_{st} in (2), one can use the staggered operator Γ_{55} for the role of γ_5 since it coincides with $\gamma_5 \otimes \mathbf{1}$ on the 2 physical flavors up to $O(a)$ effects. The significance of this is that the properties $\gamma_5^2 = \mathbf{1}$ and γ_5 -hermiticity, which are crucial in the usual overlap and domain wall constructions, continue to hold in the staggered setting with $\gamma_5 \rightarrow \Gamma_{55}$. This would not be true if one tried to use the direct staggered analogue Γ_5 of γ_5 .

The flavor-chiral mass term used in this work, which reduces the 4 staggered fermion flavors to 2 flavors, is not the only possibility. Another flavor-chiral mass term which reduces the number of flavors to 1 was subsequently proposed in [21] after the first version of this paper appeared on the Arxiv. However, it breaks more of the staggered fermion symmetries than the present proposal, namely some of the lattice rotation symmetries. The consequences of this for radiatively generated counter-terms and fine-tuning requirements remains to be seen. The 1-flavor version of staggered overlap fermions has the advantage that all the *flavored* vector and GW-chiral symmetries are exact. On the other hand, for problems where only the *unflavored* chiral symmetry is important, e.g. for computing the η' mass, the present 2-flavor formulation has advantages: it preserves lattice rotation symmetry and gives “two quarks (the u and d) for the price of one” in lattice QCD simulations.

Acknowledgments

I thank Mike Creutz, Philippe de Forcrand, Maarten Golterman and Herbert Neuberger for discussions and/or feedback. I also thank Philippe de Forcrand and his group for performing the numerical investigations in [12]

and for kindly providing me with Fig. 1. This work was supported by the AcRF Tier 1 grant RG 61/10. Some of this work was done at the National Center for Theoretical Sciences, Taiwan.

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